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Exact probability distribution for the Bernoulli-Malthus-Verhulst model driven by a multiplicative colored noise

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We report an exact result for the calculation of the probability distribution of the Bernoulli-Malthus-Verhulst model driven by a multiplicative colored noise. We study the conditions under which the probability distribution of the Malthus-Verhulst model can exhibit a transition from a unimodal to a bimodal distribution depending on the value of a critical parameter. Also we show that the mean value of $x(t)$ in the latter model always approaches asymptotically the value 1.

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In this Rapid Communication, we calculate the exact probability distribution (PD) for the Bernoulli-Malthus-Verhulst (BMV) model driven by a multiplicative colored noise. Our starting point is the stochastic differential equation (SDE)

$$\dot{x} = [a_0(t) + a_1\xi_1(t)]x(1 + bx^{\mu}), \tag{1}$$

where the deterministic growth rate $a_0(t)$ is perturbed by a colored noise $\xi_1(t)$, in which a_1, b, and μ are free parameters. Specifically we consider here $\xi_1(t)$ as the Ornstein-Uhlenbeck (OU) process defined by the SDE

$$\dot{\xi}_1(t) = -\gamma\xi_1(t) + \sqrt{c}\,\eta(t), \tag{2}$$

where $\gamma > 0$ is the reciprocal of the correlation time of the OU process and $c > 0$ is the intensity of the Gaussian white noise (GWN) $\eta(t)$, defined by the mean value $\langle\eta(t)\rangle = 0$ and the correlation function $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$. This type of model has been widely considered in the literature [1]. On setting $\mu=1$ and $b=-1$ we recover the Malthus-Verhulst (MV) or logistic model

$$\dot{x} = [a_0(t) + a_1\xi_1(t)]x(1 - x), \tag{3}$$

which was proposed many years ago for describing the survival of a population [2–5] and it is one of the most successful models in population dynamics. In addition, the MV model has found applications in many other areas of science such as social sciences [6,7], autocatalytic chemical reactions [8], biological and biochemical systems [9–11], and as an effective model for the description of the population of photons in a single mode laser [9,12,13].

Stochastic perturbations on model Eq. (3) have been considered in many places in the literature. References [2,3,5,9] are classic references. They contain several applications and developments of this model. In [14] stability conditions are obtained when the growth rate a_0 is perturbed by a GWN. In [15–18] the transient behavior has been investigated when the system is driven by the same type of perturbation and the relaxation time of the system is calculated as a function of noise intensity. In [19] the results of [16] are extended to the case in which a_0 is perturbed by a colored Gaussian noise and confirmed by an analogical experiment as well as by numerical simulations. The authors in [20] consider the model $\dot{x} = ax - bx^2 + x\xi_1(t) + \xi_2(t)$, where $\xi_1(t)$ and $\xi_2(t)$ are correlated GWNs, in order to analyze a cancer cell population. Using the Fokker-Planck equation (FPE) they analyze the behavior of the stationary PD. In [21] a connection between the logistic equation with a_0 perturbed by a GWN and the stochastic resonance in a linear system is investigated.

Next we perform the calculations for the exact PD, analyze the properties of the PD for the MV model, and evaluate the mean value of $x(t)$ providing a procedure to calculate the higher order moments for the last model.

The model Eq. (1) can be reduced to a linear SDE via the transformation

$$\xi_2 = (1/\mu)\ln[(1 + bx^{\mu})/x^{\mu}] \tag{4}$$

which leads to the equation

$$\dot{\xi}_2 = -a_0 - a_1\xi_1(t). \tag{5}$$

We emphasize that, for the points $x=0$ and $1+bx^{\mu}=0$, the transformation Eq. (4) does not hold. The behavior of the system in these points has to be analyzed through a limit procedure. The details for the case $\mu=1$ and $b=-1$ are discussed below. We now consider Eqs. (2) and (5) as a system of linear SDEs describing a Gaussian process in the variables (ξ_1, ξ_2). The corresponding FPE is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \xi_2}[(a_0 + a_1\xi_1)P] + \frac{\partial}{\partial \xi_1}(\gamma\xi_1 P) + \frac{1}{2}c\frac{\partial^2 P}{\partial \xi_1^2}. \tag{6}$$

Since in Eqs. (2) and (5) the noise is additive, the Ito and the Stratonovich prescriptions produce the same result [2,22]. To solve Eq. (6) we will use the characteristic function

$$Z(\mathbf{k},t;\xi_0,t_0) = \int e^{ik_1\xi_1 + ik_2 z}P(\xi,t|\xi_0,t_0)d\xi_1 d\xi_2. \tag{7}$$

Replacing it into Eq. (6) we obtain

$$\frac{\partial Z}{\partial t} = -ik_2 a_0 Z - ik_2 a_1\left(\frac{1}{i}\frac{\partial Z}{\partial k_1}\right) - ik_1\gamma\left(\frac{1}{i}\frac{\partial Z}{\partial k_1}\right) - \frac{1}{2}ck_1^2 Z. \tag{8}$$

Using the characteristic function for a Gaussian process [23]

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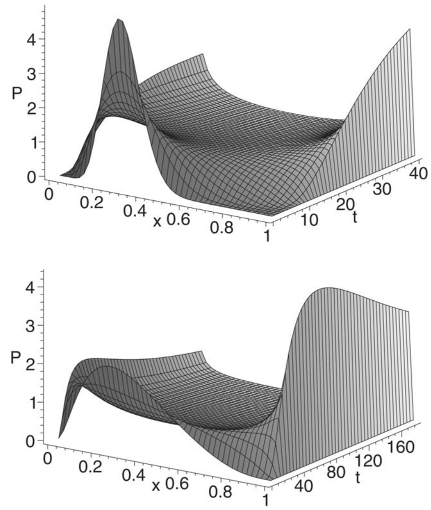

FIG. 1. Time evolution of $p(x,t)$: On the x axes the time t, on the y axes the variable x, and on the z axes $p(x,t)$. The values of the parameters are $x_0=0.3$, $a_0=0.04$, $a_1=1$, $\xi=1$, $\gamma=10$, and $c=20$. It is clear that at the beginning the probability is concentrated in a region near $x=0$ (first graphic). For large time the probability is concentrated near $x=1$ (second graphic).

$$Z(\mathbf{k},t;\xi_0,t_0)=e^{ik_\mu m_\mu(t)-(1/2)k_\mu k_\nu \sigma_{\mu\nu}(t)}, \quad (9)$$

we obtain the evolution equations for the moments $m_\mu(t)$ and for the correlation matrix $\sigma_{\mu\nu}(t)$

$$\dot{m}_1=-\gamma m_1, \quad \dot{m}_2=-a_0-a_1 m_1, \quad (10)$$

$$\dot{\sigma}_{11}=c-2\gamma\sigma_{11}, \quad \dot{\sigma}_{12}=-a_1\sigma_{11}-\gamma\sigma_{12}, \quad \dot{\sigma}_{22}=-2a_1\sigma_{12}. \quad (11)$$

The initial conditions for these equations are $m_1(t_0)=\xi_1^0$, $m_2(t_0)=\xi_2^0$, $\sigma_{11}(t_0)=0$, $\sigma_{12}(t_0)=0$, and $\sigma_{22}(t_0)=0$. The solution of the previous system (10) and (11) is

$$m_1=\xi_1^0 e^{-\gamma(t-t_0)}, \quad m_2=\xi_2^0-a_0(t-t_0)+(a_1\xi_1^0/\gamma)(e^{-\gamma(t-t_0)}-1), \quad (12)$$

$$\sigma_{11}=c/(2\gamma)(1-e^{-2\gamma(t-t_0)}), \quad \sigma_{12}=-ca_1/(2\gamma^2)(1-e^{-\gamma(t-t_0)})^2, \quad (13)$$

$$\sigma_{22}=ca_1^2/(2\gamma^3)[2\gamma(t-t_0)-(3-e^{-\gamma(t-t_0)})(1-e^{-\gamma(t-t_0)})], \quad (14)$$

where we focused on $a_0(t)=a_0=\text{const}$. Replacing Eq. (9) into Eq. (7) and taking the inverse Fourier transform we obtain

$$P(\xi,t|\xi_0,t_0)=1/\sqrt{4\pi^2\det[\sigma_{\mu\nu}]}\times\exp\{-\tfrac{1}{2}(\xi_\mu-m_\mu)\sigma_{\mu\nu}^{-1}(\xi_\nu-m_\nu)\}. \quad (15)$$

Integrating on ξ_1 gives

$$P(\xi_2,t|\xi_2^0,\xi_1^0,t_0)=1/\sqrt{2\pi\sigma_{22}}\exp\{-1/(2\sigma_{22})(\xi_2-m_2)^2\}. \quad (16)$$

Using Eq. (4) the exact PD for the process $x(t)$ is

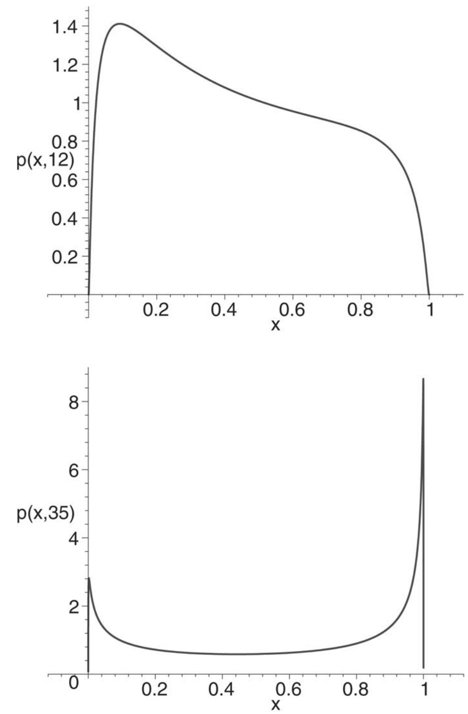

FIG. 2. PD $p(x,t)$ at two times. Values of the parameters are the same as in Fig. 1. In the first graphic $p(x,t)$ still shows a maximum, and in the second one $p(x,t)$ shows two sharp maxima near $x=0$ and $x=1$ with an evident difference in the intensity; $p(x,t)$ vanishes at the extrema of the interval.

$$P(x,t)=1/\sqrt{2\pi\sigma_{22}}1/[x(1+bx^\mu)]\times\exp\left\{-\frac{1}{2\sigma_{22}}\left[\frac{1}{\mu}\ln\left(\frac{1+bx^\mu}{x^\mu}\right)-m_2\right]^2\right\}. \quad (17)$$

Next we focus on the PD for the MV model. We study the conditions under which the PD makes a transition from a unimodal to a bimodal distribution. The PD is

$$p(x,t)=\frac{1}{\sqrt{2\pi\sigma_{22}}}\frac{1}{x(1-x)}\exp\left\{-\frac{1}{2\sigma_{22}}\left[\ln\left(\frac{1-x}{x}\right)-m_2\right]^2\right\}. \quad (18)$$

Considering the behavior at the extrema of the interval, for any finite time t, it is straightforward to show that

$$\lim_{x\to 0^+}p(x,t)=0, \quad \lim_{x\to 1^-}p(x,t)=0. \quad (19)$$

This behavior is shown in Figs. 1 and 2.

Taking the derivative of $p(x,t)$ with respect to x we find as a condition for an extremum

$$\ln(1-x)-\ln x=-2\sigma_{22}(t)x+\sigma_{22}(t)+m_2(t). \quad (20)$$

This can be solved graphically by finding the intersection between the functions $y_1=\ln(1-x)-\ln x$ and $y_2=-2\sigma_{22}(t)x+\sigma_{22}(t)+m_2(t)$ (see Fig. 3). The function y_1 is a monotonic function defined on the open interval $(0,1)$ which diverges positively for $x=0$ and negatively for $x=1$. We also note that $y_1(1/2)=0$. The straight line y_2 has a negative slope that has

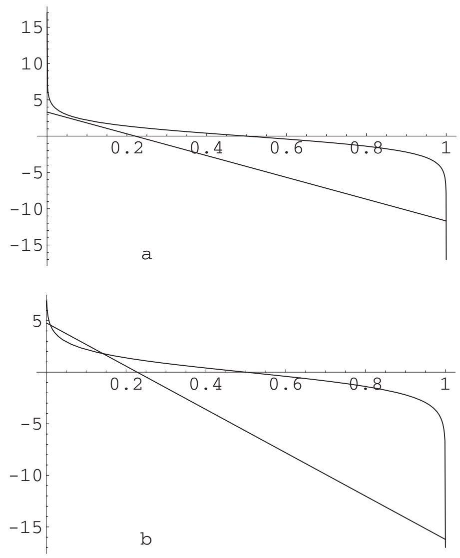

FIG. 3. Plot of $y_1=\ln(1-x)-\ln x$ and $y_2=-2\sigma_{22}(t)x+\sigma_{22}(t)+m_2(t)$. Plot (a) is for a time $t<t_c$ and plot (b) for $t>t_c$. The critical parameter satisfies the relation $a_0\tau<1$.

an intercept with the x axis at the point $\bar{x}=1/2[1+m_2(t)/\sigma_{22}(t)]$. For sufficiently large time t, $m_2(t)$ is a negative function while $\sigma_{22}(t)$ is a positive function. Hence $\bar{x}\leq 1/2$. In particular $\bar{x}\leq 0$ when $m_2(t)/\sigma_{22}(t)\leq -1$. For large time, using Eqs. (12) and (14), $\bar{x}\leq 0$ when $a_0\tau\geq 1$, where

$$\tau=\gamma^2/(ca_1^2) \tag{21}$$

is the critical parameter. In this case the straight line has only one intersection point with y_1 that asymptotically approaches the point $x=1$. The PD has only a maximum near $x=1$.

When $a_0\tau<1$ the intersection point of y_2 with the x axis lies in the interval $(0,1)$. The slope will increase in modulus up to a certain time $t=t_c$, when the slope will be the same of the tangent straight line. The equations defining the time t_c are

$$1/[(1-x_c)x_c]=2\sigma_{22}(t_c), \tag{22}$$

$$\ln(1-x_c)-\ln x_c=-2\sigma_{22}(t_c)x_c+\sigma_{22}(t_c)+m_2(t_c). \tag{23}$$

Starting from the time t_c, the straight line will have three intercepts with y_1 corresponding to two maxima, located near the points $x=0$ and $x=1$ and a minimum near the point $x=\bar{x}$; $p(x,t)$ is now a bimodal distribution (see Figs. 1 and 2). Based on Fig. 3 we can deduce that the intensity of the maximum near $x=0$ is much smaller than the intensity of the maximum near $x=1$. This is due to the fact that the intersection of the straight line with the x axis occurs at $\bar{x}<1/2$ producing this asymmetry. On the other hand, Fig. 2 shows that at large time the two maxima are very sharp approaching $x=0$ and $x=1$, respectively. A rough estimation of t_c is given by neglecting the exponential terms in $\sigma_{22}(t)$ and $m_2(t)$. Making this approximation we can find the critical time as a function of x_c from Eq. (22). It has been previously shown that the abscissa of the intersection between the straight line and the x axis, $\bar{x}$, is smaller than 1/2. As a consequence, the first intersection between the straight line and the curve must occur on the left of $\bar{x}$ (see Fig. 3). Then we can neglect the terms of order $O(x_c^2)$. Solving Eq. (22) gives

$$t_c=[3x_c(1-x_c)+\tau\gamma]/[2\gamma x_c(1-x_c)]\approx 3/(2\gamma)+\tau/(2x_c). \tag{24}$$

Substituting t_c from Eq. (24) into Eq. (23) we obtain an equation for x_c. To find an estimation of the order of magnitude of t_c we solve the equation for x_c approximately. Using Newton's method we find the final expression for the critical time

$$t_c\approx\frac{3}{2\gamma}+\frac{\tau}{2}\left[\frac{1}{4}-\frac{\frac{4}{3}-\frac{a_1\xi_1^0}{\gamma}-a_0\left(\frac{3}{2\gamma}+2\tau\right)+\ln\left(\frac{1-x_0}{3x_0}\right)}{-\frac{32}{9}+8a_0\tau}\right]^{-1}. \tag{25}$$

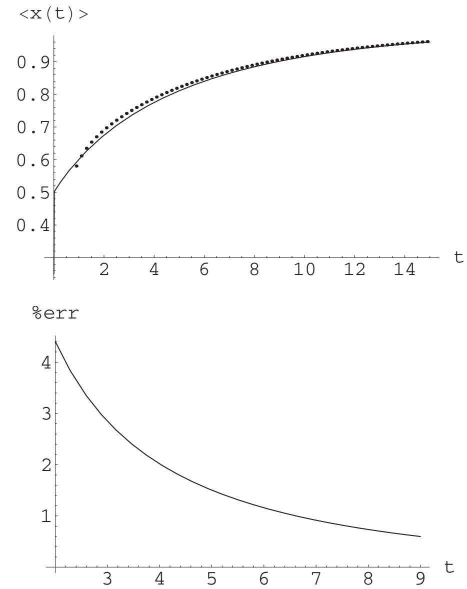

FIG. 4. Top graphic: On the abscissa the time t and on the ordinate $\langle x(t)\rangle$. The continuous line is the numerical evaluation of Eq. (26) while the dotted line is obtained using the first three terms of Eq. (28). Bottom graphic: On the abscissa the time t, starting from $t=1$, and on the ordinate the percentual error between the numerical and the analytical evaluation of $\langle x(t)\rangle$.

Using Eqs. (4) and (16), we find that the mean value of $x(t)$ is given by

$$\begin{aligned}\langle x(t)\rangle&=\frac{1}{\sqrt{2\pi\sigma_{22}(t)}}\int_{-\infty}^{\infty}\frac{1}{\exp[\xi]+1}\exp\left[-\frac{[\xi-m_2(t)]^2}{2\sigma_{22}(t)}\right]d\xi\\&=\frac{1}{\sqrt{2\pi\sigma_{22}(t)}}\left(\int_0^{\infty}\frac{\exp[-[\xi-m_2(t)]^2/[2\sigma_{22}(t)]]}{\exp[\xi]+1}d\xi\right.\\&\quad\left.+\int_0^{\infty}\frac{\exp[-[\xi+m_2(t)]^2/[2\sigma_{22}(t)]]}{\exp[-\xi]+1}d\xi\right).\end{aligned} \tag{26}$$

Under quite general conditions for $t\to\infty$ we can show that $\langle x(\infty)\rangle\geq 1/2$. Indeed

$$\langle x(t)\rangle \geq \frac{1}{\sqrt{2\pi\sigma_{22}(t)}}\int_0^\infty \frac{\exp\left[-\frac{[\xi+m_2(t)]^2}{2\sigma_{22}(t)}\right]}{\exp[-\xi]+1}d\xi$$

$$\geq \frac{1}{\sqrt{2\pi\sigma_{22}(t)}}\frac{1}{2}\int_0^\infty \exp\left[-\frac{[\xi+m_2(t)]^2}{2\sigma_{22}(t)}\right]$$

$$= \tfrac{1}{4}\{1-\mathrm{erf}[m_2(t)/\sqrt{2\sigma_{22}(t)}]\}, \quad (27)$$

where erf(x) is the error function. Imposing the condition $m_2(t)/\sqrt{2\sigma_{22}(t)} \to -\infty$ for $t\to\infty$ and using the asymptotic properties of the function erf(x) we conclude that $\langle x(\infty)\rangle \geq 1/2$. Returning to Eq. (26) and performing a series expansion for the terms in the denominator of the integral we obtain

$$\langle x(t)\rangle = \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\exp\left[-(n+1)m_2(t)+\frac{(n+1)^2}{2}\sigma_{22}(t)\right]$$
$$\times\left[1+\mathrm{erf}\left(\frac{m_2(t)-(n+1)\sigma_{22}(t)}{\sqrt{2\sigma_{22}(t)}}\right)\right]+1$$
$$-\frac{1+\mathrm{erf}\left(\frac{m_2(t)}{\sqrt{2\sigma_{22}(t)}}\right)}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2}\exp\left[nm_2(t)\right.$$
$$\left.+\frac{n^2}{2}\sigma_{22}(t)\right]\left[1-\mathrm{erf}\left(\frac{m_2(t)+n\sigma_{22}(t)}{\sqrt{2\sigma_{22}(t)}}\right)\right]. \quad (28)$$

It is worth stressing that, under the conditions $t\to\infty$, $m_2(t)\to-\infty$, and $\sigma_{22}(t)\to\infty$, the divergent exponentials in the series are balanced by the asymptotic expansion of the error function so that the convergence to the unit limit value is driven at the lowest order by the exponential $\exp[-m_2^2(t)/(2\sigma_{22}(t))]$. In general, with a few terms of the series, the percentual difference with respect to the exact result is of the order of 1% or less (see Fig. 4).

We now provide a formula to calculate higher moments. Introducing the new variable $\eta=\xi-m_2(t)$ we obtain

$$\langle x(t)\rangle = \frac{1}{\sqrt{2\pi\sigma_{22}(t)}}\int_{-\infty}^{\infty}\frac{1}{\exp[\eta+m_2(t)]+1}$$
$$\times\exp[-\eta^2/[2\sigma_{22}(t)]]d\eta. \quad (29)$$

From Eq. (29), we obtain for the nth moment of $x(t)$

$$\langle x^n(t)\rangle = (-1)^{n-1}\frac{z^n}{\Gamma(n)}\frac{\partial^{n-1}}{\partial z^{n-1}}\left[\frac{1}{z}\langle x(t)\rangle\right], \quad z\equiv\exp[-m_2(t)]. \quad (30)$$

We stress that the white noise case can be obtained in a straightforward manner from the previous results in the limit $c\to\gamma^2$ and $\gamma\to\infty$. In fact, from Eqs. (12) and (14), in this limit we obtain $m_2(t)=\xi_2^0-a_0(t-t_0)$ and $\sigma_{22}(t)=a_1^2(t-t_0)$, while for the critical parameter τ we obtain $\tau=1/a_1^2$. In the case of weak white noise, corresponding to $a_1^2t\to 0$, the Gaussian in Eq. (29) can be approximated by a Dirac delta giving a result that, unless of a scaling factor, coincides with the result of Ref. [24].

In this paper we obtained an exact solution for the PD of the BMV model driven by a multiplicative colored noise. We studied the conditions under which the PD, in the MV model, exhibits a transition from a unimodal to a bimodal distribution. This transition is regulated by a critical parameter, τ, which according to the condition $a_0\tau\geq 1$ or $a_0\tau<1$ will have a unimodal or a bimodal distribution, respectively. We performed an evaluation of the time at which this transition occurs. Next we showed that the mean value, $\langle x(t)\rangle$, always asymptotically approaches 1. For this model we provided a formula to calculate the higher order moments of the process $x(t)$. Finally we examined the white noise case. We found that the difference with the colored noise case rests on the fact that the transient terms, in the quantities m_2 and σ_{22}, vanish.

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